We start w/ a sufficient condition for a family of metric to be uniformly equivalent.

Lemma Let M^n be a closed manifold. For $0 \le t < T \le \infty$, let $g(t)$ be a 1-parameter family of metrics on M^n. If $\exists$ a constant $C < \infty$ s.t.

$$\int_0^T \left| \frac{\partial}{\partial t} g(x,t) \right|_{g(t)} dt \le C$$

$\forall x \in M^n$ then $e^{-C} g(x,0) \le g(x,t) \le e^{C} g(x,0)$ $\forall x \in M, t \in [0,T)$.

Moreover, as $t \uparrow T$, $g(t)$ converge uniformly to a continuous metric $g(T)$ s.t. $\forall x \in M$

$$e^{-C} g(x,0) \le g(x,T) \le e^{C} g(x,0).$$

Proof Let $x \in M$, $t_0 \in [0,T)$ and $V \in T_x M^n$ be an arbitrary vector. Then

$$\left| \log \left[\frac{g_{(x,t_0)}(V,V)}{g_{(x,0)}(V,V)} \right] \right| = \left| \int_0^{t_0} \frac{\partial}{\partial t} \left[\log g_{(x,t)}(V,V) \right] dt \right|$$

$$= \left| \int_0^{t_0} \frac{\partial_t g_{(x,t)}(V,V)}{g_{(x,t)}(V,V)} dt \right|$$

$$\leq \int_0^{t_0} \left| \frac{\partial}{\partial t} g_{(x,t)} \left(\frac{v}{|v|}, \frac{v}{|v|} \right) \right| dt$$

$$\leq \int_0^{t_0} \left| \frac{\partial}{\partial t} g_{(x,t)} \right|_{g(t)} dt \leq C$$

(as $|A(u,u)| \leq |A|$ for any 2-tensor A and u unit vector).

$\Rightarrow$ we get the uniform bounds by exponentiation.

$\therefore$ $|g(x,t) - g(x,T)|_{g(0)} \leq \int_t^T |\partial_t g(x,t)|_{g(0)} \to 0$ as $t \to T$

uniformly on M as M is compact and the convergence happens $\forall x \in M$.

$\therefore$ if we define the function $f: TM^n \to \mathbb{R}$ by

$$f(x,v) = \lim_{t \to T} g_{(x,t)}(v,v)$$

then this function exists and is continuous. By polarizing the equation we define the $(2,0)$ tensor $g_{(x,T)}$ on M^n w/

$$g_{(x,T)}(v,w) = \frac{1}{4} \left[f(x, v+w) - f(x, v-w) \right]$$

and hence $g_{(x,T)}(v,w) = \lim_{t \to T} g_{(x,t)}(v,w)$.

By the bounds above, we get that

$$e^{-C} g_{(x,0)} \leq f(x,v) \leq e^{C} g_{(x,0)}(v,v) \implies g(x,T) \text{ is pos-}$$

definite and $\therefore$ a continuous Riem. metric.

$\boxed{R}$

i.e. if $g(t)$ a RF satisfy uniform curvature bound on a finite time interval then all the metrics in the family are uniformly equivalent.

<u>Cor.</u> If a sol^n $g(t)$ of the RF satisfies $|Rc| \leq K$ for some constant K on $[0,T]$ then

$$e^{-2KT} g(x,0) \leq g(x,t) \leq e^{2KT} g(x,0).$$

and hence the metrics are all equivalent.

$\boxed{R}$

The main result which we want to prove in this lecture is that the curvature must explode as we approach the singular time T.

<u>Main Theorem</u> If g_0 is a smooth metric on closed M^n, then the RF $g(t)$ w/ $g(0) = g_0$ has a unique sol^n on a maximal time interval $[0,T)$ w/ $T \leq \infty$. If $T < \infty$ then

$$\lim_{t \uparrow T} \left(\sup_{x \in M} |Rm(x,t)| \right) = \infty.$$

The idea of the proof is by contradiction: If $|Rm(x,t)| \le K$ for some K then we'll prove that $g(t) \xrightarrow[\text{smooth}]{} g(T)$ w/ $g(T)$ a metric and then we can have a sol^n past T thus contradicting the maximality of T.

We have already seen above that $g(T)$ is a continuous metric. We'll now try to prove that it is actually smooth. One way to show this is to guarantee that the spatial derivatives of g near T are bounded.

Notice that the derivative estimates are completely useless at $t=0$ as we can't expect any bounds on $|\nabla^k Rm|$ from a bound on $|Rm|$. It's only after we flow the metric by the RF that the $\nabla^k Rm$'s are brought under control.

$\therefore$ for the purposes of showing smoothness of $g(T)$ we work near $t=T$, we can get derivative estimates by considering our RF as starting at some time shortly before T. We have

Cor: Let $(M^n, g(t))$ be a RF. If $\exists\ \beta, K > 0$ s.t.

$$|Rm(x,t)|_{g(t)} \le K \quad \forall\ t \in [0,T] \text{ w/ } T > \frac{\beta}{K} \text{ then}$$

$\forall$ $m \in \mathbb{N}$ $\exists$ a constant $B_m = B_m(m, n, \min\{\beta, 1\})$ s.t

$$|\nabla^m Rm(x,t)|_{g(x,t)} \leq B_m K^{1+\frac{m}{2}} \quad \forall\, t \in \left[\frac{\min\{\beta, 1\}}{K}, T\right].$$

<u>Proof</u>:- Let $\beta_1 = \min\{\beta, 1\}$. Let $t_0 \in \left[\frac{\beta_1}{K}, T\right]$. We consider the RF starting at time $T_0 = t_0 - \frac{\beta_1}{K}$. Applying the derivative estimates to this RF w/ $\alpha = \beta_1 \implies |\nabla^m Rm| \leq \frac{C_m K}{(t - T_0)^{m/2}}$

w/ $C_m = C_m(m, n, \min\{\alpha, 1\})$. $\therefore$ at $t = t_0$ we get $t_0 - T_0 = \frac{\beta_1}{K}$

$$|\nabla^m Rm| \leq \frac{C_m K}{\left(\frac{\beta_1}{K}\right)^{m/2}} = \frac{C_m}{\beta_1^m} K^{1+\frac{m}{2}}.$$

$\boxed{}$

Lemma:- Let $(M^n, g(t))$ be a RF and let $(x^i), i = 1, \dots, n$ be a local coordinate system defined on some coordinate chart $U \subset M^n$.

If $\exists$ $K > 0$ s.t.

$$|Rm(x,t)|_{g(t)} \leq K \quad \forall \; t \in [0, T)$$

then $\forall$ $m \in \mathbb{N}$ $\exists$ constants C_m, C_m' depending only on the chosen coordinate chart s.t.

$$|\partial^m g(x,t)| \leq C_m$$

and $$|\partial^m Rc(x,t)| \leq C_m' \quad \forall\ (x,t) \in \mathcal{U} \times [0,T)$$

where the norms are taken w.r.t. the Euclidean metric in (x^i).

(We are doing computations in a coordinate chart b/c we'll use the compactness of M later to have an open subcover and then use the uniform derivatives in each chart).

note that $\partial^m g$ means the expression $\partial_{i_1} \dots \partial_{i_m} g_{pq}$ in $\mathcal{U}$.

moreover Γ_{ij}^k will be treated as the coordinates of the "tensor" Γ where Γ can be taken as the difference of ∇ and $\nabla^{Eucl.}$.

$\forall\ m \in \mathbb{N}\ \exists$ a uniform upper bound on $|\nabla^m Rc|$ on $(\beta/K, T)$.

There is also an upper bound on $|\nabla^m Rc|$ on $[0, \beta/K]$ as the interval is compact $\Rightarrow$ $\boxed{|\nabla^m Rc| \leq D_m}$ $\forall\ x \in M^n,\ t \in [0,T)$, $D_m = D_m(m, g(t))$.

<u>Claim</u>:- $\exists$ constants A_m, B_m and C_m $\forall\ m \in \mathbb{N}$ s.t.

1. $|\partial^{m-1} \Gamma| \leq A_m$
2. $|\partial^m Rc| \leq B_m$ $\qquad\qquad \forall\ t \in [0,T)$.
3. $|\partial^m g| \leq C_m$

We prove the above claim by induction on m.

$m=0$ for 2) and 3) reads as $|Rc| \le B_0$, $|g| \le C_0$ which are true b/c of $|Rm| \le K$ and the uniform equivalence of the metrics respectively. $m=1$ for 1) reads as $|\Gamma| \le P_1$ which is true b/c

$$\partial_t \Gamma = g^{-1} * \nabla Rc \Rightarrow |\partial_t \Gamma| \le C \Rightarrow \Gamma \le C(T-0) = A_0.$$

Assume that 1), 2) and 3) are satisfied $\forall\, m \le p-1$ and we prove the estimates for $m=p$. We start w/ 1).

note that $|\partial^{p-1}\Gamma| \le C$ at $t=0$ as M is compact.

$$\therefore \quad \partial_t \partial^{p-1}\Gamma = \partial^{p-1}\partial_t \Gamma = \partial^{p-1}(g^{-1} * \nabla Rc)$$

$$= \sum_{i=0}^{p-1} \partial^{p-i-1}(g^{-1}) * \partial^i \nabla Rc. \quad —①$$

$\because$ $g^{ij} g_{jk} = \delta^i_k \Rightarrow$ bounds on $|\partial^k g| \Rightarrow$ bounds on $|\partial^k g^{-1}|$.

$\because$ the terms w/ derivatives on g^{-1} have order $\le p-1$ $\therefore$ we have bounds on them by the induction hypothesis. What about the other terms?

Recall that for $i \le p-1$

$$\partial^i \nabla Rc = \nabla^{i+1} Rc + \underset{\substack{0 \le j \le i-1 \\ k \le i}}{*} (\partial^j \Gamma, \partial^k Rc)$$

($\because$ for any tensor S, $\nabla S = \partial S + \Gamma * S$ and then you reiterate, e.g.,

$$\nabla^2 S = \nabla(\nabla S) = \nabla(\partial S + \Gamma * S) = \partial(\partial S + \Gamma * S) + \Gamma * (\partial S + \Gamma * S)$$
$$= \partial^2 S + \partial\Gamma * S + \Gamma * \partial S + \Gamma * \Gamma * S \text{ and so on.})$$.

$\because$ $\nabla^{i+1} Rc$ is bounded by the boxed eqn above and

$j \leq i-1 \leq p-2$, $k \leq i \leq p-1$ $\Rightarrow$ by induction we have bounds on all the other terms.

$\therefore$ $|\partial_t \partial^{p-1}\Gamma| \leq C$ for some constant C. $\because$ $\partial^{p-1}\Gamma$ is bounded at $t=0$ $\Rightarrow$ we can integrate

$$|\partial^{p-1}\Gamma| = \left|\int_0^t \partial_s \partial^{p-1}\Gamma \, ds\right| \leq \int_0^t |\partial_s \partial^{p-1}\Gamma| \, ds \leq Ct \leq C'.$$

$\therefore$ $\partial^{p-1}\Gamma$ is also bounded thus proving 1).

In a similar way

$$\partial^p Rc = \underbrace{\nabla^p Rc}_{\text{bounded}} + \underset{\substack{j \leq p-1 \\ k \leq p-1}}{*} (\underbrace{\partial^j \Gamma}, \underbrace{\partial^k Rc}_{\text{bounded by induction hypo}})$$

all terms except $\partial^{p-1}\Gamma$ bounded by induction hyp.
$\partial^{p-1}\Gamma$ is bounded as shown above.

$\Rightarrow$ $|\partial^p Rc| \leq B_p$ thus proving (2).

now, $|\partial_t \partial^p g| = |-2\partial^p Rc| \le C$ by (2) $\Rightarrow$ again integrating the bound we get $|\partial^p g| \le C_p$.

$\square$

Corr :- The limit metric $g(T)$ is smooth and $g(t) \to g(T)$ uniformly in every C^K-norm as $t \to T$.

proof :- want to prove that $g(T)$ is smooth which is same as proving that all the derivatives w.r.t. some coordinate system exist and are continuous. Take any coordinate chart U of M^n. From the RF eqn,

$$g_{ij}(x,T) = g_{ij}(x,t) - 2\int_t^T R_{ij}(x,\tau)\,d\tau \quad \forall\, t \in [0,T).$$

for any multi-index α, the previous result tells us that $\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}$ and $\frac{\partial^{|\alpha|}}{\partial x^\alpha} R_{ij}$ are uniformly bounded on $U \times [0,T) \Rightarrow$ we can differentiate under the integral sign $\Rightarrow$

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}(x,T) = \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}\right)(x,t) - 2\int_t^T \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} R_{ij}\right)(x,\tau)\,d\tau$$

$\Rightarrow \frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}(x,T)$ exists $\forall\, \alpha \Rightarrow g(T)$ is smooth.

moreover

$$\left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}(x,T) - \frac{\partial^{|\alpha|}}{\partial x^\alpha} g_{ij}(x,t) \right| \leq C(T-t)$$

$\Rightarrow$ $g(t) \to g(T)$ in any C^k-norm.

now, $\because$ M^n is compact we can choose a finite set of coordinate charts and then take the max. of all the bounds in all of the estimates.

$\boxed{\checkmark}$

Proof of the main theorem

Assume, for a contradiction, that $|Rm| = \infty$ as $t \nearrow T$ $\Rightarrow$ $|Rm(x,t)| \leq K$. $\therefore$ by the above results, $g(t)$ converge uniformly in any C^k-norm to a smooth metric $g(T)$.

$\because$ $g(T)$ is C^∞ $\Rightarrow$ by the short time existence of sol^n to the RF w/ $g(T)$ as the initial condition, we get a RF $\bar{g}(t)$ w/ $\bar{g}(0) = g(T)$ and exists for $0 \leq t < \varepsilon$.

If we consider

$$\bar{g}(t) = \begin{cases} g(t) & 0 \leq t < T \\ \bar{g}(t-T) & T \leq t < T+\varepsilon \end{cases}$$

then this is a smooth extension of the original sol^n $g(t)$ as all spatial derivatives are continuous at $t=T$. Moreover, the time derivative of quantities related to the metric are also continuous at $t=T$ b/c the time-derivative can be written in terms of the spatial derivatives which are bounded and continuous.

$\therefore$ $\bar{g}(t)$ is a R.F. w/ initial condition as $g(0)$ which can be extended past $t=T$ thus contradicting the maximality of T.

$\therefore$ $|Rm|$ must blow up as $t \nearrow T$.

$\boxtimes$

We also have the following corr.

Corr: $\rightarrow$ Suppose $(M^n, g(t))$ is a RF w/ $|Rm| \leq K$ at $t=0$. Then $\exists$ a constant $C = C(n)$ s.t.

$$|Rm| \leq \frac{K}{1 - \frac{CKt}{2}}.$$

(thus we have an upper bound on the blow-up rate of $|Rm|$).

Moreover, $\exists$ a constant $b>0$, $b = b(K,n)$ s.t. the RF exists $\forall$ $t \in [0,b)$.

Proof. The blow-up rate follows from the max. principle and the evolution of $|Rm|^2$. The existence of such a b basically follows from the proof of the doubling time estimate.

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